“Lecture notes” for Lie theory spring 2024

In these notes I will write down a plan for the course in Lie theory that is being held at Campus Førde for the robotics group and mathematicians on campus. The course is initially heavily inspired by the compendium of Brian Hall “An Elementary Introduction to Groups and Representations» that is open on arXiv, https://arxiv.org/abs/math-ph/0005032.

First session

We aim to go through some basics of abstract algebra that are required for further exploration of Lie theory.

What is a Lie group?

Let us start in medias res.

Definition (Lie Group):

A Lie group is a differentiable manifold G that also is a group such that the group operation

$$* : G \times G \to G,$$

and its inverse $g \to g^{-1}$ is differentiable.

So what is a group then?

This first session will mostly be about groups. So what is this strange and useful object?

Definition (group):

A group is a set G and an operation

$$* : G \times G \to G,$$

such that the following holds:

1. Associativity: For $g, h, i \in G$ we have $(g * h) * i = g * (h * i)$.
2. Identity: There exists an identity element $e \in G$ such that $g * e = e * g = g$ for all $g \in G$.
3. Inverse: To each element $g \in G$ there exists an inverse g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

Note that the three requirements for $(G,*)$ to be a group are called the **group axioms**. It is also implied with from $* : G \times G \to G$, that the group is closed under the operation, i.e., if $g \in G$ and $h \in G$, then $g * h \in G$.

Extra definition (Abelian group) (Yes, after the Norwegian mathematician Niels Henrik Abel)

If a group is commutative under the operation, i.e., $g * h = h * g$ for all $g, h \in G$, then the group is called an abelian group.

About the $*$

I will eventually forget to write the operation symbol $*$. Therefore, we will make the usual convention that if two letters are written next to each other it is implied that there has happened an operation between them, $g * h = gh$.

Some examples of groups

1. The **trivial** group: The set with only one element, e, is a group where the group operation is defined by $ee = e$.

"Proof that the trivial group is a group":

Let us call the group $T = \{e\}$, and check that all of the axioms are satisfied. First off all, the group is closed under the operation as $ee = e \in T$.

Associativity: $(ee)e = ee = e(ee)$.

Identity: $ee = e$ for all $e \in T$ as only e is in T.

Inverse: e is the inverse of e, therefore alle elements of T has an inverse,

2. The integers under addition, $(\mathbb{Z}, +)$: The set of integers form a group with addition $+$ as the group operation.

In-between exercise – Talk to your neighbor and convince yourself (prove) that the trivial group is a group. In other words, check that the group axioms are satisfied and that the group is closed under the group operation.

3. The real numbers $(\mathbb{R}, +)$ and real-valued vectors $(\mathbb{R}^n, +)$ under addition.
4. Nonzero real numbers under multiplication $(\mathbb{R}^+, \cdot)$.
5. Non-zero Complex numbers under multiplication $(\mathbb{C}^+, \cdot)$.
6. Complex numbers of absolute value one under multiplication S^1.
7. Invertible matrices under matrix multiplication $GL(n, \mathbb{R})$, this group is called the *general linear group.*
8. The set of matrices with determinant one is a group under matrix multiplication, $SL(n, \mathbb{R})$. This is called the *special linear group.*
9. Integers modulo n, $\mathbb{Z}_n$.
10. Permutation group. The set of one-to-one maps from $\{1,2, \dots, n\}$ onto itself is a group under function composition (not important for us, but quite fun).

Exercise break:

Split the participants into groups of two that will together prove that selections of the groups above are groups. Determine which of the groups are abelian.

Show that $\mathbb{Z}$ not is a group under multiplication.

Is $\mathbb{R}^+$ a group under the operation $a * b = \sqrt{ab}$.

Properties of groups:

1. The identity in a group is unique.
   Proof: Let G be a group and assume that there exist two elements $e \in G$ and $f \in G$ such that $eg = g = fg$ for all $g \in G$. We then necessarily have that $e = ef = f$.
2. Each element in a group has a unique inverse.
   Exercise.
3. In groups it is sufficient with $gh = e$ to be sure that h is the unique inverse of g.
   Proof: Let $g, h \in G$ be such that $gh = e$. We can now multiply both sides by the inverse of g, $g^{-1}(gh) = g^{-1}e$. Then we have (by associativity and multiplication with identity) that $h = g^{-1}$.

4. The inverse of the inverse is the element itself, $(g^{-1})^{-1} = g$.
   Exercise.

Subgroups

Definition (subgroup):

A subgroup H of a group G is a subset such that H is itself a group under the same operation as G. One only needs to check the following conditions:

1. The identity is in H.
2. If $h \in H$ then $h^{-1} \in H$.
3. H is closed, i.e., if $h_1, h_2 \in H$ then $h_1 h_2 \in H$.

Examples:

1. $\mathbb{Z}$ under addition is a subgroup of $\mathbb{R}$ under addition.
2. S^1 is a subgroup of $\mathbb{C}^+$
3. $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

Exercise: Show that this is true.

Maps between groups (homomorphisms)

We will now consider what happens when we assign elements of groups to each other using a special type of maps called a **homomorphism**.

Definition (homomorphisms):

Let G and H be groups. A map $\phi : G \to H$ is called a homomorphism if $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. Note that the group operations inside ϕ and outside ϕ are not (necessarily) the same as they can be operations on different groups.

If a homomorphism is bijective, it is called an **isomorphism**. If there exist an isomorphism between two groups, they are called **isomorphic**. Two groups that are isomorphic somehow act the same as groups, although they are not strictly the same.

An isomorphism of a group with itself is called an **automorphism**.

Proposition (Important fact in everyday normal guy's language):

Identities and inverses are preserved through homomorphisms. I.e., $\phi(e_g) = e_h$ and $\phi(g^{-1}) = (\phi(g))^{-1}$.

Proof. Exercise

Definition (Kernel):

The kernel of a homomorphism $\phi: G \to H$ is the subset $\ker(\phi) \subseteq G$ such that $\phi(g) = e_h$.

The kernel of a homomorphism is a subgroup of G. (This can easily be verified).

Proposition: If a kernel of a homomorphism only includes the identity element, then the homomorphism is injective (i.e., no two elements are sent to the same element by the homomorphism).

Proof. Assume that $\ker(\phi) = e_g$ for $\phi: G \to H$. Let now $g_1, g_2 \in G$ be such that $\phi(g_1) = \phi(g_2)$. We then have $e_h = \phi(g_1)\phi(g_1)^{-1} = \phi(g_2)\phi(g_1)^{-1} = \phi(g_2 g_1^{-1})$, i.e., $g_2 g_1^{-1} \in \ker(\phi)$. Since $\ker(\phi) = e_g$ we have that $g_2 g_1^{-1} = e_g$ and by the uniqueness of the inverse we have that $g_1 = g_2$.

(René Des)**Cartesian products:**

Let G and H be two groups. Then the Cartesian product $G \times H$ with the product $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$ is itself a group.

Both G and H are isomorphic to subgroups of $G \times H$ by fixing one element in the opposite group.

If there is more time left

We look at exercises 2, 5, 8 (what about $SL(n, \mathbb{R})$?) and 13 from the compendium.

Second session – Matrix Lie groups

In this session we will define what a matrix Lie group is (it is in some way easier to define that going through the whole process of manifolds), then we will see some examples and non-examples, look at lie group homomorphisms and finally state that all matrix Lie groups also are Lie groups (a not completely trivial result).

Definition (Matrix Lie group)

A matrix Lie group is a subgroup H of $GL(n, \mathbb{C})$ with the property that any convergent sequence in H either converges to a matrix in H or the matrix is not invertible (not in $GL(n, \mathbb{C})$).

Note 1: Convergence here, means elementwise convergence with the usual definition of limits (ϵ/δ).

Note 2: Saying that a subgroup H of $GL(n, \mathbb{C})$ is a matrix Lie group is the same as saying that H is closed in $GL(n, \mathbb{C})$. It needs however not be closed in the set of all matrices, although many are.

Note 3: To be able to talk about topological properties of Matrix Lie Groups (open/closed sets, continuous functions, differentiability) we can identify the space of $n \times n$ matrices with the space $\mathbb{C}^{n^2}$ and its usual topology (defines as one would imagine, extrapolated from 1, 2 and 3 dimensional real space).

A counter-example

A subgroup of $GL(n, \mathbb{C})$ that is not a matrix Lie group is the group of all invertible matrices with rational entries. The set is clearly a subgroup (multiplication and addition of rational numbers are still rational), but every irrational number can be written as a limit of rational numbers (a math fact), hence there are sequences of that converge in $GL(n, \mathbb{C})$ but not to elements of the subgroup. This counterexample can also be traced back to the fact that subgroups of rational numbers are not Lie groups as they cannot be manifolds (not locally similar enough to real numbers).

Examples

Here follows a list of examples of matrix Lie groups.

1. The general linear groups: $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$.

   Exercise: Prove that these are matrix Lie groups.

2. The special Linear groups: $SL(n, \mathbb{R})$, and $SL(n, \mathbb{C})$.

   Proof: Both groups are subgroups of the general linear groups. Moreover, the determinant is a continuous function, hence the limit of matrices with unit determinant also needs to have unit determinant.

3. The orthogonal and special orthogonal groups, $O(n)$, and $SO(n)$. These are subgroups of $GL(n, \mathbb{R})$.

   Reminder: Orthogonal matrices are those with orthonormal columns. The orthogonal group consist of those. The special orthonormal group is the subgroup of the orthonormal group with determinant equal to one. The proof that these are matrix Lie groups are the same as for the special linear group, i.e., the conditions to be in these groups are continuous.

   Exercise 6 in hall for classification.

4. The unitary and special unitary groups, $U(n)$, and $SU(n)$. These are the complex versions of $O(n)$ and $SO(n)$.
   Note: There is one very important distinction between the orthogonal and the unitary groups. The determinant of matrices in the unitary group has complex modulus equal to one, i.e., $|\det A| = 1$, while the matrices in the orthogonal group has determinants with "real modulus" (absolute value) equal to one. Absolute value equal to one means 1 or -1, while complex modulus equal to one governs the whole unit circle. Therefore the "size" difference between $U(n)$ and $SU(n)$ is greater than that of $O(n)$ and $SO(n)$. In fact, while considered as manifolds $U(n)$ is of one dimension more than $SU(n)$, while $O(n)$ and $SO(n)$ are of the same dimension.
   Exercise 8 in Hall.
5. The groups $\mathbb{R}^*$, $\mathbb{C}^*$, S^1, $\mathbb{R}$, and $\mathbb{R}^n$ can all be shown to be isomorphic to matrix Lie groups.
   Exercise: Prove it! Hint: The groups they are isomorphic to have all been defined above.
6. The group of nonzero quaternions $\mathbb{H}$.

   It is isomorphic to the subgroup of $GL(2, n)$ with matrices of the form $\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}$ with isomorphism

$$\phi(a + bi + cj + dk) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

   Exercise: Should we try to show that the group of nonzero quaternions indeed is a group? Furthermore, that the isomorphism actually is an isomorphism?

7. The group of unit quaternions are isomorphic to $SU(2)$.
   The same isomorphism as the above does the trick. In other words, if we understand the matrix Lie group $SU(2)$ we can also understand the group of unit quaternions that can act on three-dimensional space to produce any rotation.
8. The Euclidean and the special Euclidean groups $E(n)$, $SE(n)$. The Euclidean group is the group of all distance preserving bijections of $\mathbb{R}^n$ to itself. It also turns out that any element in $E(n)$ is of the form

$$f(\vec{v}) = R\vec{v} + \vec{t},$$

   where $R \in O(n)$ and $\vec{t} \in \mathbb{R}^n$. For this reason, the Euclidean group is also often called the group of Rigid transformations. It does however not preserve orientations, and for that reason the special Euclidean group might be even more often considered. It is denoted

$SE(n)$, and often called the group of rigid body motions (or just the group of rigid motions) and is the subgroup of $E(n)$ with $R \in SO(n)$.

Exercise:

1. Show that $E(n)$ is a group with function composition as the operation.
2. $E(n)$ is not, directly, a matrix Lie group as it is not a subgroup of $GL(n, \mathbb{C})$. Find a suitable subgroup of $GL(n+1, \mathbb{C})$ and an isomorphism showing that $E(n)$ (and thereby also $SE(n)$) is isomorphic to a matrix Lie group, you then also need to argue that this subgroup of $GL(n+1, \mathbb{C})$ has the closedness property that is required to be a matrix Lie group.
   Secret hint: consider matrices of the form
   $$\begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix}.$$

Definition (Lie group homomorphism)

A homomorphism between two matrix Lie groups G and H is a ***Lie group homomorphism*** if the homomorphism is continuous. If the homomorphism additionally is an isomorphism and the inverse map is continuous then it is called a ***Lie group isomorphism*.**

Theorem: All matrix Lie groups are Lie groups.

Third session

The third session will be a proper exploration of rotations in 2D ($S^1, SO(2)$), rotations in 3D quaternions, $SU(2)$, $S^3, SO(3)$, and of the rigid body motions.

A vague proof that $SU(2)$ is twice as big as $SO(3)$.

Fourth session – The exponential map

The session starts with some work on the Rigid motions (the exercises above). Then we explore the unit circle and its Lie algebra before we delve into the theory on exponential maps.

We are now going to turn our attention towards what is known as the Lie algebra, which is a linear space (a vector space) that comes as a pair with any Lie group, with an additional product structure (called the bracket). The way that we are going to define this might potentially be a bit different as to how we are going to think about it, but it will probably still be quite useful to see it done in this way. At least it helped be, in comparison to pure handwaving[1].

The goal here will be to define the Lie algebra as the space that maps into a matrix Lie group through the exponential map. Therefore, we begin with the exponential map, and a small trip down memory lane. Recall (one of) the definition(s) of the Euler number

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2{,}718281828.$$

The first definition should probably be

[1] Handwaving is mathematical terminology for speaking/writing about things while not writing them down with proper mathematical notation and proofs. Unfortunately, the definition of being handwavy is itself handwavy, and it might vary significantly from mathematician to mathematician what is considered to be handwavy and what is considered to be rigorous (the opposite). Moreover, me mentioning handwaving above is pure nonsense as most mathematician will call this entire note for handwaving, and there is no way we are going to do Lie algebras rigorously.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

A crazy number with some properties that are found to be useful in all of mathematics. Naturally, in Lie theory as well. One of these properties is that treating the Euler number as the base of exponentiation as a function lead to the exponential function which is defined for real and complex numbers as

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

The exponential function can be defined on any set of objects that can be operated (multiplied) together in the same way. Specifically, we can do it with matrices X:

$$\mathrm{Exp}(X) = \sum_{n=0}^{\infty}\frac{X^n}{n!}.$$

It is also customary to write $e^X = \mathrm{Exp}(X)$, but I am a bit sceptic to use that notation as it feels like all rules of powers should hold, and that is not entirely true. Therefore, in this text, I will only use $\mathrm{Exp}(X)$ and it should be understood as the series presented above.

Here, we will consider the exponential function on matrices, as we are concerned with matrix Lie groups. First, a list of properties of the exponential function $\mathrm{Exp}(X)$ will be presented.

Proposition:

The following statements hold for the matrix exponential function:

1. The matrix exponential is well-defined for all (square) matrices, i.e., the series $\sum_n^\infty \frac{x^n}{n!}$ converges for all square matrices X. In fact, the series is *absolutely convergent* (even stronger).
2. $\mathrm{Exp}(0) = I$.
3. $\mathrm{Exp}(X)$ is invertible and $\mathrm{Exp}(X)^{-1} = \mathrm{Exp}(X^{-1})$.
4. $\mathrm{Exp}\big((\alpha+\beta)X\big) = \mathrm{Exp}(\alpha X)\mathrm{Exp}(\beta X)$ for all $\alpha, \beta \in \mathbb{R}$.
5. $\big(\mathrm{Exp}(X)\big)^m = \mathrm{Exp}(mX)$ for any $m \in \mathbb{N}$.
6. If $XY = YX$ (X and Y commutes), then $\mathrm{Exp}(X+Y) = \mathrm{Exp}(X)\mathrm{Exp}(Y) = \mathrm{Exp}(Y)\mathrm{Exp}(X)$.
7. If C is invertible then $\mathrm{Exp}(CXC^{-1}) = C\mathrm{Exp}(X)C^{-1}$.
8. $||\mathrm{Exp}(X)|| \leq \mathrm{Exp}(||X||)$.
9. $\det(\mathrm{Exp}(X)) = \mathrm{e}^{\mathrm{trace}(X)}$ (Think a bit about how crazy this is, the left is a nightmare to compute, the right is trivial.)
10. $\frac{d}{dt}\mathrm{Exp}(tX) = X\mathrm{Exp}(tX) = \mathrm{Exp}(tX)X$, for a parameter $t \in \mathbb{R}$.

Notice that 3., 4., and 5., all are results of 6.

How do we compute the exponential map of a matrix

The practical answer is to be clever and find a formula. The general answer is the following:

Let X be any square matrix. Then it follows from the Jordan canonical form that X can be written as the sum

$$X = S + N,$$

where S is diagonizable ($S = CDC^{-1}$, with D diagonal) and N is nilpotent ($N^m = 0$, for some $m \in \mathbb{N}$, and hence for all $\ell \geq m$). Moreover, $SN = NS$. Therefore, by property 5 above

$$\mathrm{Exp}(X) = \mathrm{Exp}(S + N) = \mathrm{Exp}(S)\mathrm{Exp}(N).$$

We can in other words compute the exponential of any matrix if we can compute it for diagonalizable and nilpotent matrices. For diagonalizable matrices we use property 6 above, and for nilpotent matrices we can compute it explicitly since the series expansion terminate after $n = m$.

Extra properties of the exponential map

As with the "standard" definition of Euler's number, or the exponential function for real numbers, we have a similar one for the Exponential function for matrices

$$\mathrm{Exp}(X) = \lim_{n\to\infty} \left(I + \frac{X}{n}\right)^n.$$

There is also a formula, called "Lie's formula" that states that

$$\mathrm{Exp}(X + Y) = \lim_{m\to\infty} \left(\mathrm{Exp}\left(\frac{X}{m}\right)\mathrm{Exp}\left(\frac{Y}{m}\right)\right)^m.$$

The logarithmic function

The exponential function also has an inverse function, namely the logarithmic function. This one is, as with all inverse functions, endowed in some mystery. However, we can make a general definition here as well through the following theorem.

Theorem:

The function

$$\mathrm{Log}(A) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - I)^m}{m}$$

is well-defined and continuous for all square matrices with complex entries with $||I - A|| < 1$. Moreover, $\mathrm{Log}(A)$ is real if A is real and

$$\mathrm{Exp}\big(\mathrm{Log}(A)\big) = A,$$

for A with $||I - A|| < 1$, and

$$\mathrm{Log}\big(\mathrm{Exp}(X)\big) = X,$$

for $||X|| < \log 2$.

Note that due to property 8 about the exponential function we have $||I - \mathrm{Exp}(X)|| < 1$ whenever $||X|| < \log 2$.

Session 5 – The Lie algebra

The Lie algebra $\mathfrak{g}$ of the matrix Lie group G is the set of all matrices X such that

$$\mathrm{Exp}(tX) \in G, \forall t \in \mathbb{R}.$$

The Lie algebras of some Lie groups:

1. The general linear groups $GL(n,\mathbb{C}), GL(n,\mathbb{R})$. For X to be in the Lie algebra of $GL(n,\mathbb{C})$ we need $\exp(tX) \in GL(n,\mathbb{C})$ for all $t \in \mathbb{R}$. However, by property 3 for the exponential function this is true for all complex $n \times n$ matrices X. This Lie algebra is denoted by $\mathfrak{gl}(n,\mathbb{C})$. The same holds for $GL(n,\mathbb{R})$ and the Lie algebra is then called $\mathfrak{gl}(n,\mathbb{R})$.
2. The special linear groups $SL(n,\mathbb{C}), SL(n,\mathbb{R})$. As above the requirement for the exponential of X to be invertible always holds, but here we also require the determinant of the exponential to be 1, $\det(\mathrm{Exp}(tX)) = 1$ for all $t \in \mathbb{R}$. This is governed by property 9 for the exponential map, i.e.,
$$\det(\mathrm{Exp}(tX)) = e^{\mathrm{trace}(tX)} = e^{\mathrm{t\cdot trace}(X)} = 1.$$
This implies that $t \cdot \mathrm{trace}(X) = 0$ for all $t \in \mathbb{R}$, which again implies that $\mathrm{trace}(X) = 0$.

   In other words, $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{R})$ consists off all $n \times n$ matrices with zero trace.
3. See the appendix for the Lie algebra of the special unitary group.
4. The Euclidean groups: $E(n)$ and $SE(n)$. **Exercise.**

Session 6 – More on the Lie algebra

We start by discussing the Lie algebra of the special unitary group. Then we have as an exercise to compute the one for the Euclidean group.

Properties of the Lie algebra.

1. If $X \in \mathfrak{g}$ then $sX \in \mathfrak{g}$ for all $s \in \mathbb{R}$.
2. If $X, Y \ \in \mathfrak{g}$ then $X + Y \in \mathfrak{g}$.
3. If $X, Y \ \in \mathfrak{g}$ then $XY - YX \in \mathfrak{g}$.

"Proof":

The first is true by the definition of the Lie algebra. The second one is true by the Lie product formula, and the third is more complicated.

Definition (bracket/commutator):

The map
$$[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
defined by $[A, B] = AB - BA$ is called the bracket or the commutator and has the following properties:

1. It is bilinear (linear in each component).
2. It is skew-symmetric $[A, B] = -[B, A]$.
3. It upholds the Jacobi identity: $\big[A, [B, C]\big] + \big[C, [A, B]\big] + \big[B, [C, A]\big] = 0$.

Exercise: What is the bracket operation on $\mathfrak{su}(2)$? **Hint**: It very closely resembles a well-known product.

General Lie algebras

It is common to define Lie algebras without the stick necessity of an underlying Lie group. A Lie algebra is in that regard a vector space, $\mathfrak{g}$, together with a bilinear (bracket) map $[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with the following properties

1. Skew-symmetry: $[A, B] = -[B, A]$.
2. Jakobi identity: $\big[A, [B, C]\big] + \big[C, [A, B]\big] + \big[B, [C, A]\big] = 0$.

The first property also implies that $[A, A] = 0$.

Lie algebra homomorphisms

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A Lie algebra homomorphism is a linear map $\tilde{\phi}$ between $\mathfrak{g}$ and $\mathfrak{h}$ such that

$$\tilde{\phi}([A, B]) = [\tilde{\phi}(A), \tilde{\phi}(B)].$$

The induced Lia algebra homomorphism

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let now ϕ be a Lie group homomorphism from G to H. Then, there exists a unique Lie algebra homomorphism $\tilde{\phi}$ such that

$$\phi(\mathrm{Exp}(X)) = \mathrm{Exp}\left(\tilde{\phi}(X)\right),$$

Moreover,

a) $\tilde{\phi}(AXA^{-1}) = \phi(A)\tilde{\phi}(X)\phi(A)^{-1}$ for all $A \in G$, and $X \in \mathfrak{g}$.
b) $\tilde{\phi}(X) = \frac{d}{dt}|_{t=0}\phi\left(\left(\mathrm{Exp}(tX)\right)\right)$, for all $X \in \mathfrak{g}$.

As a result, the Lie algebras of isomorphic Lie groups are isomorphic. And the isomorphisms can be computed using b). The converse is only sometimes true:

Theorem:

If G and H are simply connected, $\mathfrak{g}$ and $\mathfrak{h}$ are their related Lie algebras (respectively), and $\tilde{\phi}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$, then $\tilde{\phi}$ is induced by a homomorphism from G to H.

Exercise:

Consider:

- $SO(2)$, $O(2)$ and $\mathbb{R}$.
- $SU(2), SO(3)$ and $O(3)$.

Compute their Lie algebras (with brackets) and discuss them in relation to the above theorem.

Appendix and other thoughts

Here, a range of topics will be written down until they eventually, possibly, find their way into the main text.

About the unit circle and its group structure

The unit circle (for example described through the complex numbers)

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is a group under multiplication $\cdot$. Let us check that the closedness and the axioms for being a group are satisfied:

First, remember that all elements in S^1 can be described by an angle through Euler's formula

$$z \in S^1 \Rightarrow \exists \theta \in \mathbb{R} \text{ such that } z = e^{i\pi\theta}.$$

Now, given three arbitrary elements of S^1, $z_1 = e^{i\pi\theta_1}$, $z_2 = e^{i\pi\theta_2}$ and $z_3 = e^{i\pi\theta_3}$, we can first multiply two of them together to check that the group is closed under multiplication

$$z_1 \cdot z_2 = e^{i\pi\theta_1} \cdot e^{i\pi\theta_2} = e^{i\pi(\theta_1+\theta_2)} \in S^1.$$

Then we have the three axioms:

1. Associativity:

$$(z_1 \cdot z_2) \cdot z_3 = \left(e^{i\pi\theta_1} \cdot e^{i\pi\theta_2}\right) \cdot e^{i\pi\theta_3} = e^{i\pi(\theta_1+\theta_2)} \cdot e^{i\pi\theta_3} = e^{i\pi(\theta_1+\theta_2+\theta_3)} = e^{i\pi\theta_1} \cdot \left(e^{i\pi\theta_2} \cdot e^{i\pi\theta_3}\right) = z_1 \cdot (z_2 \cdot z_3).$$

2. Identity: $z = 1 = e^0$.
3. Inverse: Given $z = e^{i\pi\theta}$, the inverse is given by $z^{-1} = e^{-i\pi\theta}$.

And what is a differentiable (smooth) manifold?

That is a bit more work to define, but we will manage.

A manifold is a (topological) space that locally is equivalent (homeomorphic) to Euclidean space ($\mathbb{R}^n$). Many fancy words, but the important property is that around every point on the manifold there should exist a neighborhood and an invertible continuous map (whose inverse is also continuous) from that neighborhood to a subset of $\mathbb{R}^n$.

Sidenote on topological spaces

A topological space is a set equipped with (rules for how to define) open sets (closed sets are complements of the open sets). In that regard S^1 is a topological space where all sets of the form $(z_1, z_2) \coloneqq \left\{z = e^{i\pi\theta} \in S^1 \mid \theta \in (\theta_1, \theta_2)\right\}$ (and unions of them) are open sets.

The unit circle S^1 is a manifold

We can easily check that S^1 is a manifold. We already saw that it is a topological space. Now we just need to see that it is locally homeomorphic to Euclidean space. Given a point $z \in S^1$, we can always choose a small neighborhood (a small open set as defined above, important that the angles are less than π apart) around the point $(z_1, z_2) = \left(e^{i\pi\theta_1}, e^{i\pi\theta_2}\right)$.

About unit quaternions, SU(2), and the Lie algebra

The Lie algebra for $SU(2)$ is denoted by $\mathfrak{su}(2)$, and are found by using the characteristics of being a matrix in $SU(2)$.

Recall; a matrix is in $SU(2)$ if, and only if, it is Unitary ($U^*U = I$) and its determinant is one. In other words, an element X is in its Lie algebra $\mathfrak{su}(2)$ if

$$\operatorname{Exp}(tX)^* \operatorname{Exp}(tX) = I$$

and

$$\det(\operatorname{Exp}(tX)) = 1$$

for all $t \in \mathbb{R}$.

Differentiating the first expression and setting $t = 0$ we see that

$$X^* = -X.$$

In the other direction we see that

$$U^*U = I \Leftrightarrow U^* = U^{-1},$$

and therefore

$$\text{Exp}(tX)^* = \text{Exp}(tX)^{-1}.$$

By theorem … this is equivalent to

$$\text{Exp}(tX^*) = \text{Exp}(-tX).$$

Therefore, we also have the other direction that if $X = -X^*$ then $\text{Exp}(tX)^* \, \text{Exp}(tX) = I$.

A matrix of the form $X^* = -X$ is called skew-Hermitian (Hermitian is the complex conjugate word for symmetric).

The other requirement, that $\det(\text{Exp}(tX)) = 1$ is resolved by the theorem saying that

$$\det(\text{Exp}(X)) = e^{\text{trace}(X)}.$$

Now, if $\det(\text{Exp}(tX)) = 1$, then $\mathrm{e}^{(\text{trace}(tX))} = \mathrm{e}^{(\mathrm{t}\cdot\text{trace}(X))} = 1$ which implies that

$$t \cdot \text{trace}(X) = 2\pi i \cdot n$$

For all $t \in \mathbb{R}$ and (possibly differing $n \in \mathbb{Z}$). This can only happen if $\text{trace}(X) = 0$.

Hence $\mathfrak{su}(2)$ is the vector space of 2×2 matrices that are skew-Hermitian and have zero trace. It is easily verifiable that these matrices have the form

$$\begin{pmatrix} bi & c + di \\ -c + di & -bi \end{pmatrix}$$

For $b, c, d \in \mathbb{R}$. We can then make a basis for the Lie algebra using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the basis for $\mathfrak{su}(2)$ is given by

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{and } i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is also VERY interesting to note that as any quaternion $a + bi + cj + dk$ can be written on the matrix form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

the pure quaternions and the vector space $\mathfrak{su}(2)$ are isomorphic as algebras (they function exactly the same in terms of addition, multiplication and multiplication by scalar). This equivalence will be made important know when constructing the exponential map from $\mathfrak{su}(2)$ to $SU(2)$. Moreover, the space $\mathfrak{su}(2)$ is isomorphic to $\mathbb{R}^3$ as vector spaces.

Constructing the exponential map

The exponential map takes any element in $\mathfrak{su}(2)$, which we 2now can write as a general pure quaternion as $p = bi + cj + dk$, and sends it to $SU(2)$. It functions as

$$\exp(p) = \sum_{n=0}^{\infty} \frac{p^n}{n!}.$$

We will now do some clever tricks. First of all, a pure quaternion can be written as

$$p = \theta\,\vec{v}$$

Where $\vec{v}$ is just the normalized version of p, i.e., $\vec{v} = \frac{bi+cj+dk}{\sqrt{b^2+c^2+d^2}}$ and $\theta = \sqrt{b^2+c^2+d^2}$.

Notice now the exponentiation pattern for unit pure quaternions:

$$\vec{v}^0 = 1, \vec{v}^1 = \vec{v}, \vec{v}^2 = -1, \vec{v}^3 = -\vec{v}, \vec{v}^4 = 1, \ldots$$

Hence, we can write the exponential function as

$$\mathrm{Exp}(p) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\vec{v}^n = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + \vec{v}\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = \cos(\theta) + \sin(\theta)\,\vec{v}\,.$$

Note how this gives us a unit quaternion (i.e., an element of $SU(2)$ for all pure quaternions).

Constructing the logarithmic map

The logarithmic map should be the inverse of the exponential map, i.e., $\log(\mathrm{Exp}(p)) = p$ and $\mathrm{Exp}(\log(q)) = q$.

Written down carefully, we would want it to take a unit quaternion

$$q = a + bi + cj + dk,$$

and make it into versor form

$$q = a + r\vec{v},$$

Where $r = \sqrt{b^2+c^2+d^2}$, and $\vec{v} = \frac{(bi+cj+dk)}{r}$.

Then it should find the angle θ, such that $a = \cos(\theta)$ and $r = \sin(\theta)$. Notice that this angle always will be in one of the first two quadrants as $r \geq 0$. The angle can then be found by the rule

$$\theta = \begin{cases} \arcsin(r)\,, a \geq 0 \\ \pi - \arcsin(r)\,, a < 0 \end{cases}.$$

By doing so, we guarantee that

$$q = \cos(\theta) + \sin(\theta)\,\vec{v},$$

and we can define $\log(q) := \theta\vec{v}$.